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Consideration to the Correction Factor for Single-Dish Observations with the ACA Correlator

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Seiji Kameno	JAO/NAOJ	26 October 2014

Consideration to the Correction Factor for Single-Dish Observations with the ACA Correlator (draft)

Seiji Kameno

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This report theoretically addresses the non-linearity of total power (TP) measurements of single-dish observations using the ACA correlator (ACAC). We review the digital signal processing in the digitizers and the ACA correlator. The non-linear relation between input signal power and total power measurements with the correlator will be discussed. For continuum sources, the theoretically derived correction factor is 1.256 that meets the observational implication of 1.28 ± 0.04. In the case of spectral lines, the correction factor depends on the line width relative to the bandwidth. For narrow spectral lines, the theoretical correction factor is 1.14. The empirical correction factor of 1.206 ± 0.04 is slightly different from the theoretical implication.

1 Introduction

The single-dish observations with the TP (total power) array in ALMA is carried out using the ACA correlator. Accuracy of flux measurements in the TP array is crucial to obtain high-quality images combined with the interferometric observations of the Morita array and the main array. However, it is acknowledged that the single-dish flux measurements using the ACA correlator is affected by non-linearity, which sometimes achieve at $\sim 50\%$ error (CSV-2880), and has been the blocker of single-dish SBs of Cycle-1/2 observations.

The Total Power Non-Linearity Tiger Team has worked to investigate the characteristics of the non linearity and have quested for a remedy to correct the non-linearity. Verification observations toward continuum sources and emission line sources have carried out, as logged in JIRA CSV-2880 and CSV-3079. Statistical analyses of these tests concluded the correction factor of 1.28 and 1.2 to be multiplied to continuum and spectral line observations, respectively. The conclusion of the TP Tiger Teem was to apply a single correction factor of 1.25 to meet the tentative requirement of 10% accuracy for Cycle 1 and 2.

Besides empirical determination of the correction factor, theoretical reasoning was desired (1) to explain the necessity of the correction factor to the community, (2) to identify the scope and the application range of of the correction factor, and (3) to quest further accuracy targeting 1%.

In this report we attempt to derive the non linearity and the correction factor inductively.

Table 1: Symbols used in this report

$\hat{V}_j$	digitized voltage at $t = j\Delta t$ where Δt is the sampling period
v_h	threshold voltages ($h = -3, -2, \ldots, +3$)
q	3-bit quantized levels ($q = 0, 1, \ldots, 7$)
w_q	weight at the q-th quantization level
P_q	probability at the q-th quantization level
$C(\tau)$	analog correlation function of the time lag, τ
$\hat{C}_j$	digitized correlation function at $\tau = j\Delta t$
$s(\nu)$	complex voltage spectrum as a function of the baseband frequency, ν
$S(\nu)$	analog power spectrum
$\hat{S}_k$	digitized power spectrum at the frequency of $\nu = k\Delta\nu$ where $\Delta\nu$ is the channel spacing

Table 2: Constants used in this report

Symbol	Value	Remarks
v_{th8}	0.586	The optimal v_{th}/σ value to maximize the 3-bit quantization efficiency
v_{th16}	0.335	The optimal v_{th}/σ value to maximize the 4-bit quantization efficiency
F_8	0.7964	The 3-bit quantized power sensitivity at the optimal level, derived by $\dfrac{\partial \hat{C}_0}{\partial C(0)}$
F_{16}	0.9123	The 4-bit re-quantized spectral sensitivity at the optimal level
η_8	0.9625	The 3-bit quantization efficiency for cross correlation at the optimal level, derived by $\left(\dfrac{\partial \hat{\rho}}{\partial \rho}\right)_{\rho=0}$

1.1 Basic Theory on Spectrum and Correlation Function

Let $V(t)$ the analog baseband signal to input to the digitizer, and $s(\nu)$ is the complex voltage spectrum as the Fourier transform of $V(t)$ as

$$s(\nu) = \lim_{T\to\infty} \frac{1}{T} \int_{t=-T/2}^{T/2} V(t) e^{-i2\pi\nu t} dt. \tag{1}$$

$$\tag{2}$$

Hereafter, expression of the Fourier transform is simplified as $s(\nu) = \mathrm{FT}\left[V(t)\right]$. The power spectrum is given as $S(\nu) = |s(\nu)|^2$.

According to the Wiener–Khinchin theorem, the power spectrum relates to the correlation function, $C(\tau) = \langle V(t)V(t-\tau)\rangle$, as[1].

$$S(\nu) = \mathrm{FT}\left[C(\tau)\right], \tag{3}$$

$$C(\tau) = \mathrm{FT}^{-1}\left[S(\nu)\right]. \tag{4}$$

The correlation function at zero-lag is proportional to the power, W, as $C(0) = \left\langle V^2 \right\rangle = ZW$, where Z is the impedance of the signal transmission. This is also identical to the integral of the power spectrum as $C(0) = \int_{\nu=-B}^{B} S(\nu)\ d\nu$, where B is the bandwidth. When the probability density function of $V(t)$ follows the zero-bias normal distribution, $N(\mu = 0, \sigma^2)$ with the variance of σ^2, we have $\sigma^2 = \left\langle V^2 \right\rangle = C(0)$.

[1] Since $V(t)$ is real, $s(\nu)$ is an Hermite function. Thus, $S(\nu)$ is an even real function. In some physical description where frequency is limited in positive values, the power spectrum is folded as $\tilde{S}(|\nu|) = 2S(|\nu|)$.

2 Digital Spectral Measurements

In this section, we formularize the power spectral measurements in 3-bit quantized digital signals. We employ the zero-bias normal distribution of input analog voltage. The sampling period is at the Nyquist rate of $\Delta t = 1/(2B)$.

2.1 3-bit Quantization of Voltages

We assume that the threshold voltages are uniformly spaced as $v_h = hv_{\mathrm{th}},\ h = -3, -2, \ldots, +3$. In the ACAC, the quantized voltages, w_q, at 8 levels ($q = 0, 1, \ldots, 7$) are coded as $w_q = q - 3.5$.

The power spectrum of digitized signal is calculated via FFT (Fast Fourier Transform) in the ACAC The time-series data, $\hat{V}_j = w_q(t - j\Delta t)$ will be converted to spectrum via FFT and then complex multiplication will be take to produce the digitized power spectrum, $\hat{S}_k$, where k stands for the spectral channel. The digitized correlation function, $\hat{C}_j$ relates to the digitized power spectrum via FFT as,

$$\hat{C}_j = \frac{1}{N} \sum_{k=0}^{N-1} S_k \exp\left[i \frac{2\pi kj}{N}\right], \tag{5}$$

where j stands for the time lag and N is the number of spectral channels.

2.2 Relation between Analog and Digital Correlation Coefficient

Since quantization is a non-linear operation to reduce amount of information in the received signal, the digitized correlation function is distorted from the original analog correlation function. The digitized power spectrum is indeed modified from the original one, too.

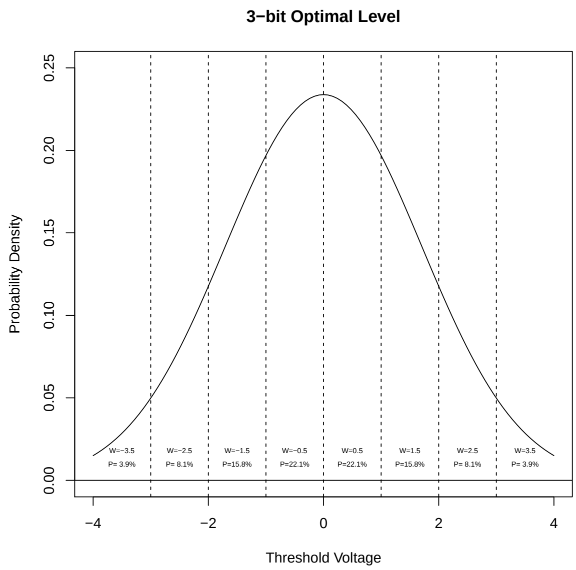

Figure 1: The probability density of input analog voltages at the optimal with respect to the threshold voltages. The vertical dashed lines indicate the threshold voltages. The standard deviation, σ is set at $\sigma = 1/v_{\mathrm{th}8} = 1.706$. Digitized code (weight) and probability at each quantized level are also shown onset.

2.2.1 Zero-Lag Correlation Coefficient

The zero-lag value of the digitized correlation function, $\hat{C}_0$ is given by

$$\hat{C}_0 = \langle w_q^2 \rangle = \sum_{q=0}^{7} P_q w_q^2. \tag{6}$$

Here, P_q is the probability where a sample states at the q-th quantization level.

For the normal distribution of analog voltages, the probabilities of quantized levels are given as

$$P_q = \mathrm{erf}\left(\frac{v_{q-3}}{\sigma\sqrt{2}}\right) - \mathrm{erf}\left(\frac{v_{q-4}}{\sigma\sqrt{2}}\right), \tag{7}$$

where $v_{\pm 4} = \pm\infty$. Figure 1 shows the probability of quantized level, P_q, for the case of $v_{\rm th}/\sigma = v_{\rm th8} = 0.586$.

Figure 2a shows the relation between analog power and digital power measurements. Both zero-lag values, $C(0)$ and $\hat{C}_0$ are scaled to be unity at the optimal level of $v_{\rm th}/\sigma = v_{\rm th8}$.

For position-switching observations toward a continuum source, the antenna temperature, T_a is calculated as

$$T_a = \frac{C(0)_{\rm on} - C(0)_{\rm off}}{C(0)_{\rm off}} T_{\rm sys}, \tag{8}$$

$$\hat{T}_a = \frac{\hat{C}_{0,\rm on} - \hat{C}_{0,\rm off}}{\hat{C}_{0,\rm off}} T_{\rm sys}, \tag{9}$$

where $T_{\rm sys}$ is the system noise temperature. The ratio of $\hat{T}_a$ to T_a is given by the derivative of $\hat{C}_0$, i.e., $\frac{\partial \hat{C}_0}{\partial C(0)}$, if $T_a \ll T_{\rm sys}$. Figure 2b shows the derivative, $\frac{\partial \hat{C}_0}{\partial C(0)}$. The derivative is $F_8 = 0.7964$ at the optimal level, which corresponds to the correction factor of $\left(\frac{\partial \hat{C}_0}{\partial C(0)}\right)^{-1} = 1.256$.

2.2.2 Correlation Coefficients at Non-Zero Lag

Correlation coefficients at lags other than 0, $C(\tau \neq 0)$ are usually much smaller than $C(0)$. The relation between the digitized correlation coefficient and analog one is known as the Van Vleck relation. Since the Van Vleck relation is established in the normalized correlation coefficient, $\rho(\tau) = \frac{C(\tau)}{C(0)}$, we argue about the normalized correlation coefficient and then multiply the zero-lag correlation coefficient if necessary.

Figure 3(a) shows the Van Vleck relation of 3-bit quantization at the optimal level. By definition, the digitized normalized correlation coefficient, $\hat{\rho}$ takes 0 and 1 at ρ =0 and 1, respectively.

The slope of $\hat{\rho}$ at $\rho = 0$ is $\left(\frac{\partial \hat{\rho}}{\partial \rho}\right)_{\rho=0} = 0.9626$, which is also known as the quantization coefficient. For small correlation coefficients, $\hat{\rho}$ is approximately proportional to ρ with the slope of the quantization coefficient. Figure 3(b) shows the departure of $\hat{\rho}$ from the linear regression. The departure is negligible for $\rho < 0.2$ and less than 1% for $\rho < 0.9$.

In the case of white noise, the correlation function will be the delta function where $C(\tau \neq 0) = 0$. For bandwidth-limited white noise, the correlation function will be the sinc function as $C(\tau) = C(0)\frac{\sin 2\pi B\tau}{2\pi B\tau}$, and the discrete sampling of the correlation function with the sampling period of $\Delta\tau = 1/(2B)$ will result in $C(j\Delta\tau) = 0$ $(j \neq 0)$.

Non-flat bandpass shape or presence of line spectrum causes non-zero values of $C(\tau \neq 0)$. Nevertheless, these values are usually small, compared with $C(0)$. Figure 4 shows the real power spectra and the correlation function. The correlation coefficient at $\tau \neq 0$ is indeed much smaller than the unity.

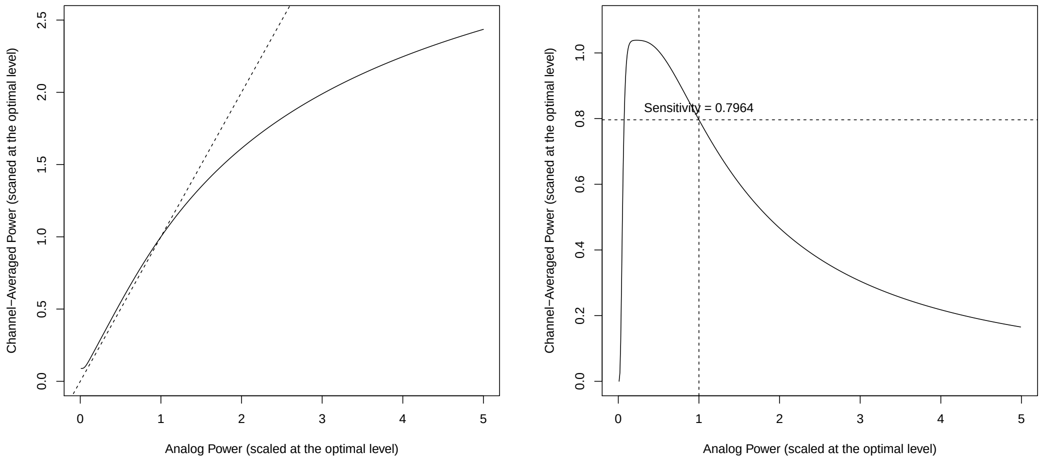

Figure 2: (a):Relation between analog power, $C(0)$ and corresponding digital power, $\hat{C}_0$. Both values are scaled at the optimal input level. The digital power measurements become significant underestimation than the ideal linear relation (dashed diagonal line) at greater power input than the optimal level. (b):The derivative of digital power, $\dfrac{\partial \hat{C}_0}{\partial C(0)}$. At the optimal level, the derivative is $F_8 = 0.7964$ that yields the correction factor of $1/F_8 = 1.256$.

In summary, the correlation coefficient at non-zero lag is approximately given as $\hat{C}_{j\neq 0} = \hat{C}_0 \hat{\rho} = \eta_8 \hat{C}_0 \rho = \eta_8 \dfrac{\hat{C}_0}{C(0)} C(\tau \neq 0) = \eta_8 C(\tau \neq 0)$.

2.2.3 4-bit Re-Quantization in Spectra

In the ACAC, 4-bit re-quantization is took place for both real and imaginary parts of complex voltage spectrum, $s(\nu)$. This introduces additional non-linearity, especially in autocorrelation.

We assume that the complex voltage spectrum follows the normal distribution. Then, we can recall similar arguments of §2.2.1 for the relation between power spectra before and after the re-quantization.

Figure 5 shows the relation between non-re-quantized and re-quantized power spectra at a spectral channel. The derivative of re-quantized power spectra at the optimal power level is $F_{16} = 0.9123$, which corresponds to the correction factor of $1/F_{16} = 1.0961$.

Note that the correction factor is spectral-channel based unlike the zero-lag value with 3-bit quantization. The channel-by-channel power was scaled into a flat-spectrum at optimal power level. The scaling factor is determined through calibration scans before every sub scans. Thus, the 4-bit non-linearity correction factor should be applied only for difference from the calibration scan, i.e., line features. Alternatively, it is identical to apply the correction factor to the correlation function except zero lag.

3 Correction Factor for Single-Dish Observations

Now we argue the correction factor to be applied to the autocorrelation power spectrum generated in the ACAC.

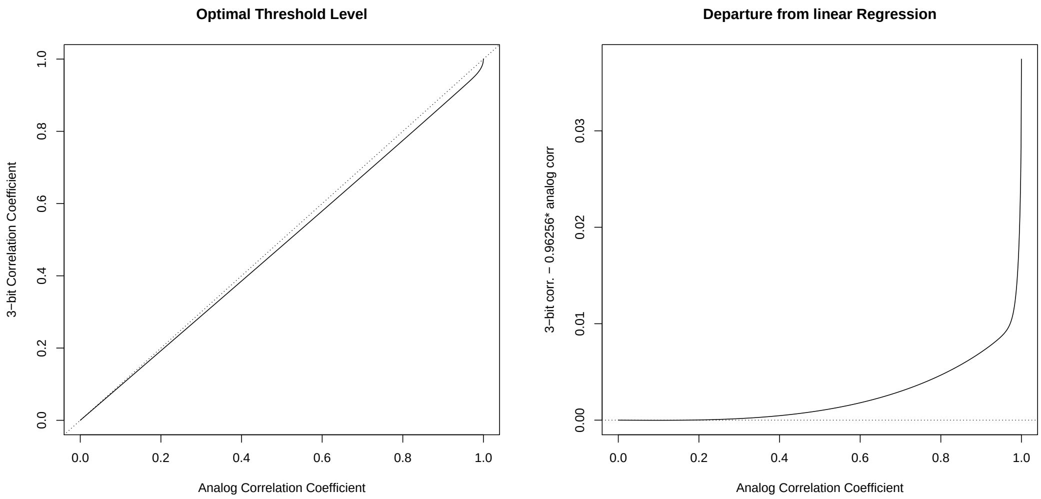

Figure 3: Relation between correlation coefficient in analog (horizontal axis) and 3-bit quantized (vertical axis) measurements.

3.1 Continuum

Recall the discussion in §2.2.1, the zero-lag correlation coefficient of 3-bit digitized signals is affected by non linearity and its sensitivity at the optimal level is $F_8 = 0.7964$. Thus, we should apply the correction factor of $1/F_8 = 1.256$ to the on-off power. This value is consistent with the empirical factor of 1.28 for continuum, reported in CSV-2880/CSV-3079.

3.2 Line

In the case of spectral line sources, we have to apply different correction factors considering its spectral shape. We discuss in the correlation function in lag domain because of different quantum efficiencies influenced on the zero-lag and non-zero-lag values.

The increment of zero-lag correlation coefficient in 3-bit quantization is F_8 times the un-quantized case. On the oter hand, the correlation coefficients at non-zero-lag values are multiplied by the quantum efficiency, η_8 for 3-bit quantization. Furthermore, the 4-bit re-quantization reduces the on–off spectral power by $1 - F_{16}$ Thus, the correlation function will be

$$\hat{C}^{\rm on}_j - \hat{C}^{\rm off}_j \quad = \quad \begin{cases} F_8 \left(C^{\rm on}(0) - C^{\rm off}(0) \right) & (j = 0) \\ \eta_8 F_{16} \left(C^{\rm on}(j\Delta\tau) - C^{\rm off}(j\Delta\tau) \right) & (j \neq 0) \end{cases} \tag{10}$$

Figure 6 shows the effect of 3-bit and 4-bit quantizations to a simulated emission-line spectrum. The correlation function of quantized signals (red) is affected from the un-quantized one (black) as expressed in equation 10. The intensity of corresponding spectrum is reduced by the factor of $\eta_8 F_{16}$. On the other hand, the efficiency in the zero-lag value affects reduction in power spectrum in the band uniformly. Therefore, the ratio of quantized line intensity to un-quantized is given as $\eta = \eta_8 F_{16}(1-b) + F_8 b$, where b is the ratio of effective line width to the bandwidth. Thus, the line correction factor, c is dependent on the line width as $c = 1/\eta = \dfrac{1}{\eta_8 F_{16}(1-b) + F_8 b}$. Figure 7 plots the line correction factor as a function of line width.

The correction factor at the narrowest end of the line profile is $1/(\eta_8 F_{16}) = 1.139$ and approaches to $1/F_8 = 1.256$ at the widest line width (i.e. continuum). There exist a small difference from the deductive implication of 1.2 for line emission reported in CSV-2880. The line width is less than 3% of

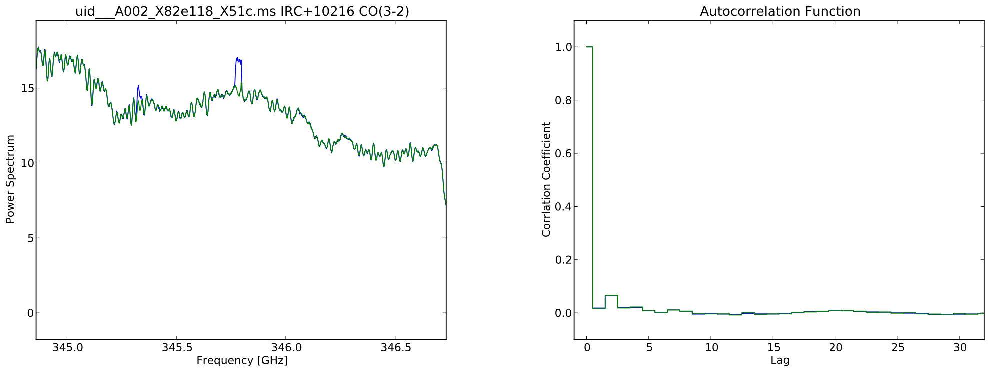

Figure 4: (a): Power spectra of on-source (blue) and off-source (green) toward IRC+10216. CO (3-2) line emission can be seen together with the bandpass shape. (b): Normalized autocorrelation functions of on-source (blue) and off-source (green). The correlation coefficient at $\tau \neq 0$ is less than 0.1.

the bandwidth and the correction factor of 1.14 was expected. At this moment it is unclear to explain the difference between theoretical and observed correction factors was 5.7%.

4 Conclusions

We reviewed the digital signal processing in the digitizers and the ACA correlator, and argued the non-linearity of quantized signals to evaluate the correction factor for single-dish observations.

For continuum observations, the theoretical correction factor is 1.256. The verification results of 1.28 ± 0.04 was consistent with the theoretical implication.

In the case of spectral line observations, the correction factor depends on the line width relative to the bandwidth. For narrow spectral line, the theoretical correction factor is 1.14. The observational correction factor of 1.206 ± 0.04 was slightly different from the theoretical implication, hence, it is necessary to clarify the cause of the difference to quest better accuracy of 1%.

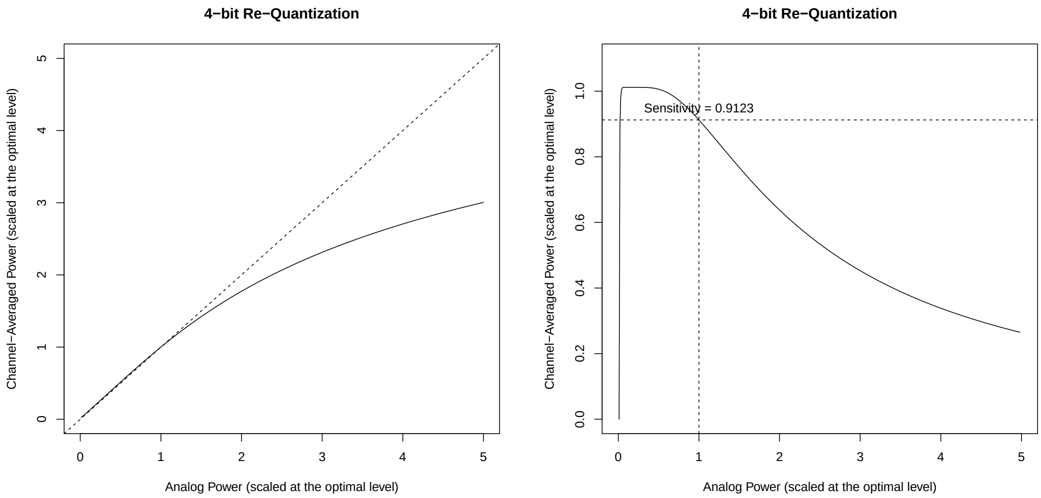

Figure 5: (a):Re-quantized power versus un-re-quantized power at a spectral channel. Both powers are scaled at the optimal level. (b):The derivative of the re-quantized power spectra. At the optimal level, the derivative is $F_{16} = 0.9123$.

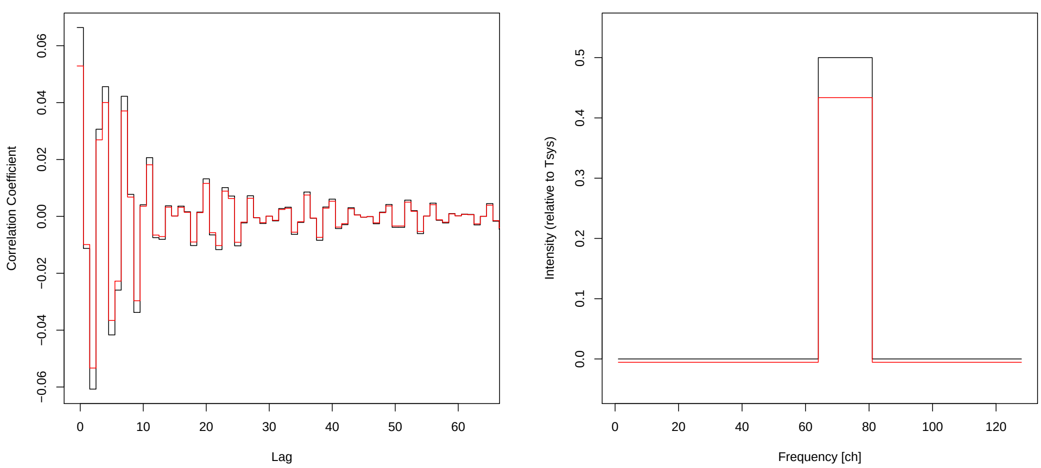

Figure 6: (a): On - Off correlation function of simulated line emission. Solid black line indicates correlation function of unquantized signal. The red line shows after 3-bit quantization + 4-bit re-quantization; quantum efficiencies for $\hat{C}_0$ and $\hat{C}_{j\neq 0}$ are F_8 and $\eta_8 \times F_{16} = 0.8781$, respectively. (b): Simulated line profile (black) and quantized profile (red).

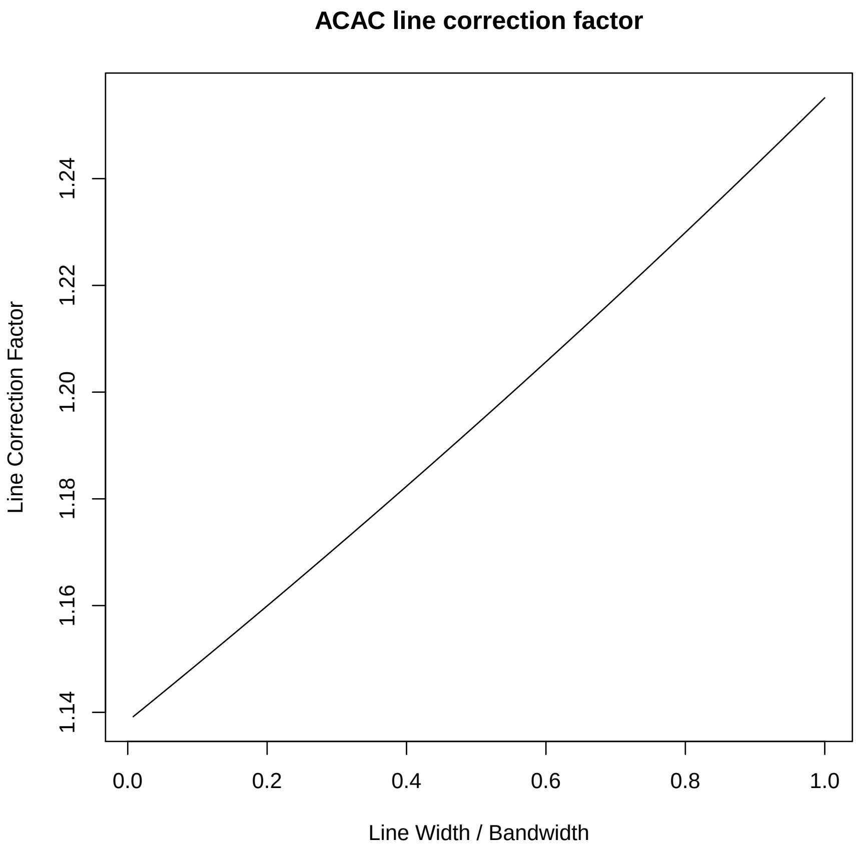

Figure 7: Correction factor as a function of line width relative to the bandwidth.